Grade 11/12 Math Circles

March 1 2023

Dynamical Systems and Fractals - Problem Set

1. Consider the function $f(x) = x^2$. Sketch this function and plot the first few points of its orbit $\{x_0, x_1, x_2, x_3, \ldots\}$, i.e. plot the points $(x_0, x_1 = f(x_0))$, $(x_1, x_2 = f(x_1))$, etc..., for the starting values $x_0 = 0, 1/2$, and 2. Describe what is happening to the orbit of $f(x)$ for each of these starting values.
2. Let $f(x) = x^2 + 3x + 1$. Find all of the fixed points of $f(x)$.
3. Consider the family of functions defined by $f_c(x) = cx$ where c is a constant and $c \neq 0$. Determine all of the fixed points of $f_c(x)$.

   Hint: You may end up with different fixed points depending on the value of c.
4. (a) Consider the function $f(x) = x^2 - \frac{1}{2}$. Sketch $f(x)$ and $y = x$ on the same set of axes and show graphically that $f(x)$ has two fixed points. Label these fixed points on your sketch as $\bar{x}_1$ and $\bar{x}_2$ such that $\bar{x}_1 < \bar{x}_2$.

   (b) Use a graphical method (i.e. cobweb diagram) to help determine the behaviour of various orbits starting near both $\bar{x}_1$ and $\bar{x}_2$. Use your diagram to make an educated guess as to the nature (attractive, repelling, or neither) of each fixed point.

   (c) Now consider the family of functions $f_c(x) = x^2 + c$ where c is a constant. For what values of c do fixed points of $f_c(x)$ exist? Some sketches of the graphs of $f_c(x)$ for various values of c may help, but they are not necessary.
5. Let $f(x) = -x^3$. Find all fixed points and periodic points of period two of $f(x)$.
6. **CHALLENGE** Let $f(x) = 1 - x^2$. Find all fixed points and periodic points of period two of $f(x)$.
7. **CHALLENGE** Consider the function $f(x) = x + \cos(x)$. Show that $f(x)$ has an infinite number of fixed points.